University of Bahrain

College of Information Technology
Department of Computer Science

ITCS253 Discrete Structures II

Second Semester 2014/2015

Final Exam – Two Hours

************ Key Solution ************

STUDENT NAME	**** **Key Solution** ****
STUDENT#	**** **Key Solution** ****
SECTION	**** **Key Solution** ****
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This exam contains **5 pages** (including this cover page) and **7 questions**. Check to see if any pages are missing. Enter all requested information on the top of this page.

You are allowed to use Calculators.

You *are not allowed* to use books, notes, or mobiles

Question	Points	Score
1	7	
2	7	
3	8	
4	8	
5	6	
6	6	
7	8	
Total:	50	

Instructor: Dr. Ali Alsaffar   Sections# 1 & 2

Answer all questions

(1) (a) **[4 points]** Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $f(x) = |x+2| - 1$. Find the range of f. Show your work.

> **Solution:** Let $y = f(x) = |x+2| - 1$,
> $$\therefore y = \begin{cases} x+1, & \text{if } x+2 \geq 0 \\ -x-3, & \text{if } x+2 < 0 \end{cases} \implies y = \begin{cases} x+1, & \text{if } x \geq -2 \\ -x-3, & \text{if } x < -2 \end{cases}$$
> For $x \geq -2 \implies y = x+1 \implies x = y-1 \implies y-1 \geq -2 \implies y \geq -1$.
> For $x < -2 \implies y = -x-3 \implies x = -y-3 \implies -y-3 < -2 \implies y > -1$.
> Hence, the range is $\{y \in \mathbf{R} \mid y \geq -1\}$.

(b) **[3 points]** Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $f(x) = 5x^2 + 2x + 1$. Is $f(x)$ one-to-one? Show your work.

> **Solution:** Let $f(x_1) = f(x_2)$
> $\implies 5x_1^2 + 2x_1 + 1 = 5x_2^2 + 2x_2 + 1$
> $\implies 5(x_1^2 - x_2^2) = -2(x_1 - x_2)$
> $\implies (x_1 - x_2)(x_1 + x_2) = (-\frac{5}{2})(x_1 - x_2)$
> Either $(x_1 = x_2)$ or $(x_1 \neq x_2)$.
> If $x_1 = x_2$ we are done.
> If $x_1 \neq x_2$, then $(x_1 - x_2) \neq 0$ and
> $x_1 + x_2 = -\frac{7}{2}$, which is impossible since $x_1, x_2 \in \mathbf{Z}$.
> Thus, $x_1 = x_2$ and f is one-to-one.

(2) (a) **[3 points]** What is the *characteristic equation* of $a_n = 3a_{n-1} + (3^n - 1)(3^n + 1)$

> **Solution:**
> $$\begin{aligned} a_n - 3a_{n-1} &= (3^n)^2 - 1 \\ &= (3^2)^n - 1 \\ &= 9^n - 1 \end{aligned}$$
> $\therefore$ The characteristic equation is $(x-3)(x-9)(x-1) = 0$.

(b) **[4 points]** Consider the operation $*$ on $\mathbf{R}$ given by $a * b = a + b + ab$. Find the identity element e and show that there is no inverse for -1.

> **Solution:** *Identity Element.*
> Let e be the identity element. Then,
> $e * a = a \implies a + e + ae = a \implies e(1+a) = 0 \implies e = 0$
> $a * e = a \implies e + a + ea = a \implies e(1+a) = 0 \implies e = 0$.
> *Inverse of -1.* Let a^{-1} be the inverse of a. Since $e = 0$, then
> $$a * a^{-1} = e \implies a + a^{-1} + aa^{-1} = 0 \implies a^{-1}(1+a) = -a \implies a^{-1} = \frac{-a}{a+1}.$$
> $\therefore (-1)^{-1} = \dfrac{-(-1)}{-1+1} = \dfrac{1}{0} \notin \mathbf{R}$. Hence, the inverse of -1 does not exists.

(3) Answer the following questions. Each question is independent.

(a) **[1 point]** Is it possible for a tree to have all vertices of degree 2? Justify your answer.

> **Solution:** No. Because a tree must have at least one leaf.

(b) **[1 point]** Why the below two trees are not isomorphic?

Solution: In the first graph the vertex v_4 is of degree 3 (the only vertex) and is adjacent to only one vertex of degree 1 (leaf). However, the vertex w_2 in the second graph is also of degree 3 but adjacent to vertices of degree 1 (leaves.)

(c) **[3 points]** Suppose G is a simple graph with degree sequence $1, 2, 3, 3, 3, 4$. Draw the graph.

Solution:

(d) **[3 points]** Consider a tree T with n number of vertices. If T has six vertices of degree 2, eight vertices of degree 3, and the remaining vertices are leaves. Find the number of vertices n and the number of edges e.

Solution: Using the degree equation

$$6 \times 2 + 8 \times 3 + (n - 14) = 2(n - 1) \implies 36 - 14 + 2 = 2n - n \implies n = 24.$$

(4) Answer the following questions. Each question is independent.

(a) **[2 points]** Let $T(1) = 1,\ T(n) = 3T(n/2) + 1$ be a recurrence relation of an algorithm. Transform $T(n)$ into inhomogeneous linear recurrence relation.

Solution: Let $n = 2^k,\ k = 0, 1, 2, \ldots$. Then,
$T(2^k) = 3T(2^k/2) + 1 \implies T(2^k) = 3T(2^{k-1}) + 1 \implies T(2^k) - 3T(2^{k-1}) = 1.$

(b) **[6 points]** Solve the recurrence relation $a_0 = 0,\ a_1 = 6$, and $a_n = 11a_{n-1} - 28a_{n-2}$ using the homogeneous technique.

Solution: $a_n - 11a_{n-1} + 28a_{n-2} = 0$
The characteristic equation is $x^2 - 11x + 28 = 0 \Rightarrow (x - 7)(x - 4) = 0$
Thus the roots are $r_1 = 7, r_2 = 4$.
$\therefore a_n = c_1(7)^n + c_2(4)^n$
From boundary conditions:

$a_0 = 0 \to 0 = c_1 + c_2$
$a_1 = 6 \to 6 = 7c_1 + 4c_2$.

Solving for c_1 and c_2 we get $c_1 = 2$ and $c_2 = -2$
$\boxed{\therefore a_n = 2 \cdot 7^n - 2 \cdot 4^n}$.

(5) **[6 points]** Answer the following questions. Each question is independent.

(a) How many odd integers in the range 10000-99999 such that no digit is repeated.

Solution: $5 \times 8 \times 8 \times 7 \times 6$

(b) How many ways a six people $ABCDEF$ sit around a a table so that A and B are next to each other.

> **Solution:** $2 \times (5-1)! = 2 \times 4!$

(c) How many numbers in the range 1–300 are divisible by 3 but not 5.

> **Solution:** Let A and B be the sets of those numbers between 1 and 300 that are divisible by 3 and 5 respectively. Then, $|A| = \left\lfloor \frac{300}{3} \right\rfloor = 100, |A \cap B| = \left\lfloor \frac{300}{3 \times 5} \right\rfloor = 20.$
> $\therefore |A - B| = |A| - |A \cap B| = 100 - 20 = 80.$

(d) Suppose x_1, x_2, and x_3 are nonnegative integers. How many solutions are there to the equation $x_1 + x_2 + x_3 = 15$.

> **Solution:** Let $n = 3, r = 15$. The number of solutions is
> $$\binom{n+r-1}{r} = \binom{3+15-1}{15} = \binom{17}{13} = 2380$$

(6) [**6 points**] In how many ways can two couples A and B form a line so that

(a) the A couple are beside each other?

> **Solution:** Let N be the set of lines in which the A can stand beside each other and M be the set of lines in which the B can stand beside each other. The A couple has 2 choices to stand next to each other and if A is considered as one unit there are 3! possible lines when standing with the others.
> $$|N| = 2 \times 3! = 12$$

(b) each couple is together?

> **Solution:** Couple A have 2 choices to stand next to each other and similarly for couple B. If the couples are considered as two units there are 2! ways: $\therefore |N \cap M| = 2 \times 2 \times 2! = 8$.

(c) at least one couple is together?

> **Solution:**
> $$\begin{aligned} |N \cup M| &= |N| + |M| - |N \cap M| \\ &= 12 + 12 - 8 \\ &= 16 \end{aligned}$$

(d) exactly one couple is together?

> **Solution:**
> $$\begin{aligned} |N \oplus M| &= |N \cup M| - |N \cap M| \\ &= 16 - 8 \\ &= 8 \end{aligned}$$

(7) (a) [**3 points**] Find r if $P(5, r) = 2P(6, r-1)$.

Solution:

$\implies P(5,r) = 2P(6,r-1)$

$\implies \dfrac{5!}{(5-r)!} = 2 \cdot \dfrac{6!}{6-r+1)!}$

$\implies \dfrac{5!}{(5-r)!} = 2 \cdot \dfrac{6 \times 5!}{(5-r+2)!}$

$\implies \dfrac{1}{(5-r)!} = \dfrac{12}{(5-r+2)(5-r+1)(5-r)!}$

$\implies (5-r+2)(5-r+1) = 12.$

Let $x = 5-r \implies (x+2)(x+1) = 12 \implies x^2+3x-10=0 \implies (x+5)(x-2)=0.$

Either $x=-5 \implies 5-r=-5 \implies \boxed{r=10}$ or $x=2 \implies 5-r=2 \implies \boxed{r=3}$.

Since $r \leq 5$, then $r=3$.

(b) **[2 points]** What is the coefficient of $a^2b^3c^2d^5e^4$ in the expansion of $(a+2b-3c+2d+e)^{16}$.

Solution:

$$\frac{16!}{2!3!2!5!4!} \cdot 2^3 \cdot (-3)^2 \cdot 2^5$$

(c) **[3 points]** Find the coefficient of the term x^{15} in the expansion of $\left(x + \dfrac{1}{x}\right)^{21}$.
Show your work using the *Binomial Theorem.*

Solution: $(x+x^{-1})^{21} = \sum_{k=0}^{21} \binom{21}{k} \cdot x^{21-k} \cdot (x^{-1})^k = \sum_{k=0}^{21} \binom{21}{k} \cdot x^{21-2k}$.

Let $x^{15} = x^{21-2k} \implies 21-2k=15 \implies 6=2k \implies k=3.$
Hence, the coefficient of x^{15} is

$$\binom{21}{3} = 1330$$